TWMS J. Pure Appl. Math., V.2, N.1, 2011, pp.146-150

ALGEBRAIC PROPERTIES OF GENERIC SINGULARITIES

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Abstract. Let $X \subset \mathbb{P}^n$ be a nonsingular projective variety of dimension r over an algebraically closed field k which is appropriately embedded if $\mathrm{char}(k) \neq 0$. By a result of Joel Roberts, there exists a projection $\pi : X \longrightarrow \mathbb{P}^m$ from a linear center onto $X' = \pi(X)$ where $r + 1 \leq m \leq 2r$, such that most of the singularities of X' are of specific parametric form, and these projections are generic. Following some known results, we give the local defining ideals of these singularities at points where X' is analytically irreducible. Under a convenient specialization, the local defining ideal of X' at any such point turns out to be a square-free monomial ideal. We revisit some algebraic properties of the associated simplicial complexes. We also give a depth formula at such points which leads to a partial affirmative answer to a conjecture of Andreotti-Bombieri-Holm on the weak normality of X'.

Keywords: generic projections, generic singularities.

AMS Subject Classification: 14B05, 13F55.

1. Introduction

A local classification of algebraic varieties, without passing to the completion, is almost inaccessible. The Cohen structure theorem on complete regular local rings over a field, is the best classification one could hope for the completion of the local rings at nonsingular points of algebraic varieties. The next step in the local classification is to consider "the most simple" singularities. A generic projection of a nonsingular projective variety into a lower dimensional projective space, is a natural procedure to produce singularities of rather simple structure, the generic singularities. For curves, this was examined in the mid 19th century, they are just nodes (see [4, Ch. IV, Theorem 3.10]). The first example of an analytically irreducible generic singularity on a surface is a "pinch" point resulting from a generic projection of a smooth surface in $\mathbb{P}^4$ onto a surface in $\mathbb{P}^3$ which was discovered by Max Noether in 1888 [2]. A general theory in higher dimension has been developed by several authors such as Holme [5], Kleiman [6], Lluis [7], Mount and Villamayor [11], Mather [8] and Roberts [12], [13]. One approach to study such singularities, is to arrange a suitable scheme structure for a morphism of algebraic varieties, and prove that in the case of generic projections, these schemes are either empty or smooth [13, §12]. This resulted to certain explicit canonical forms for most of the generic singularities. Such a treatment for differentiable mappings has previously been established by Thom [15], Morin [10] and Boardman [1]. The present article is based on the canonical forms of generic projections obtained by J. Roberts in [13]. We will only review purely algebraic properties of these singularities through [13], [14], [16] and [19].

2. Generic projections, generic singularities and their local defining ideals

Let $X \subset \mathbb{P}^n$ be a projective variety of dimension r over an algebraically closed field k. Let $L \subset \mathbb{P}^n \setminus X$ be a linear subspace with $\dim(L) = d$ given by linear equations

$$L : \ell_1 = \cdots = \ell_{n-d} = 0.$$

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Manuscript received September 2009.

A linear projection $\pi = \pi_L$ with center L is defined by

$$\pi : X \longrightarrow \mathbb{P}^m,$$

$$\pi(P) = [\ell_1(P) : \cdots : \ell_{n-d}(P)],$$

where $m = n - d - 1$. The map π is a finite morphism. The center L could be considered as a point in $\mathbf{Gr}(n, d)$, the Grassmann variety of d-dimensional linear subspaces of $\mathbb{P}^n$. A property of π is said to be generic if it holds for projections with center $L \in U$ for some open everywhere dense subset $U \subset \mathbf{Gr}(n, d)$. For example, generic projections are birational morphisms.

We now assume that X is nonsingular. Furthermore, if $\mathrm{char}(k) \neq 0$, we assume that $X \subset \mathbb{P}^n$ is appropriately embedded (see [13 §9]). Roberts has shown that there exists a linear subspace L such that for the projection $\pi = \pi_L$, and for $y = \pi(x)$ outside a closed subset of $X' = \pi(X)$ of codimension $2(m - r + 2)$ in X', the corresponding homomorphism

$$\pi^* : \widehat{\mathcal{O}}_{\mathbb{P}^m, y} \longrightarrow \widehat{\mathcal{O}}_{X, \pi^{-1}(y)}$$

has a canonical form in the sense of [13, Theorem 12.1] (or see [16 §2]). Moreover, such projections are generic. Accordingly, the singularities of X' are called *generic singularities*.

In this article we will be restricted to the points y where X' is analytically irreducible, i.e., $\pi^{-1}(y) = \{x\}$. We also assume that y has multiplicity $q \geq 2$, i.e.,

$$\dim_k \mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x} = q,$$

where $\mathfrak{m}_y$ is the maximal ideal in $\mathcal{O}_{X',y}$. For convenience we may assume that $q - 1 | r - 1$ and $m = r + \frac{r-1}{q-1} - 1$ (see [14 §1]). In this case, identifying $\widehat{\mathcal{O}}_{\mathbb{P}^m, y}$ with $R = k[[z_i, u_{ij} : 0 \leq i \leq m - r, 1 \leq j \leq q - 1]]$ and $\widehat{\mathcal{O}}_{X,x}$ with $B = k[[t, u_{ij} : 0 \leq i \leq m - r, 1 \leq j \leq q - 1]]$, π^* can be identified with the canonical forms

$$\pi^*(z_0) = u_{01} t + u_{02} t^2 + \cdots + u_{0,q-1} t^{q-1} + t^q,$$

$$\pi^*(z_i) = u_{i1} t + u_{i2} t^2 + \cdots + u_{i,q-1} t^{q-1}, \quad 1 \leq i \leq m - r,$$

and π^* is the identity map on the rest of indeterminates. Then

$$\widehat{\mathcal{O}}_{X',y} = \widehat{\mathcal{O}}_{\mathbb{P}^m, y} / \ker \pi^*.$$

The ideal $\ker \pi^*$ is said to be the *local defining ideal* of X' at y.

In the case of smooth mappings, the same canonical forms as above were given by Morin [10]. Observe that $R[t]/(z_0 - \pi^*(z_0))$ is a free R-module with basis $1, t, \cdots, t^{q-1}$.

Proposition 2.1. *(see [14, Theorem 4.11] and [16, §2]). With the notation above, put*

$$f_i(t) = z_i - \pi^*(z_i), \quad i = 0, 1, \cdots, m - r.$$

Let

$$\psi_i : R[t]/(f_0(t)) \longrightarrow R[t]/(f_0(t))$$

be the multiplication by f_i for $i = 1, \cdots, m - r$. Let M_i be the matrix of ψ_i with respect to the basis $1, t, \cdots, t^{q-1}$. Let

$$\mathcal{M} = [M_1 M_2 \cdots M_{m-r}].$$

Then the local defining ideal of X' at y is generated by the maximal minors of $\mathcal{M}$.

3. Certain specializations and the associated simplicial complexes

Let $\mathcal{M}_0$ be the matrix obtained from $\mathcal{M}$ by the following specialization

$$u_{01} = \cdots = u_{0,q-1} = 0.$$

Then it turns out that

$$\mathcal{M}_0 = [C_1 C_2 \cdots C_{m-r}],$$

where C_i is a $q \times q$ *circulant* matrix with entries equal to $\pm$ distinct indeterminates (in each row) and for $i \neq j$ the set of indeterminates appearing in C_i and the set of those appearing in C_j are disjoint (see [17]).
If $\mathrm{char}(k) \nmid q$, since k is algebraically closed, each matrix C_i is diagonalizable over k. This leads to the following result.

Proposition 3.1. [19, §4]. *Let $R_0 = k[z_i, u_{ij} : 1 \leq i \leq m-r, 1 \leq j \leq q]$. Then*

$$R_0/I_s(\mathcal{M}_0) \cong S/I_s(\mathcal{N}),$$

where

$$\mathcal{N} = [N_1 N_2 \cdots N_{m-r}],$$

$$N_i = \begin{bmatrix} y_{i1} & 0 & 0 & \cdots & 0 \\ 0 & y_{i2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots & \\ 0 & 0 & 0 & \cdots & y_{iq} \end{bmatrix},$$

and $S = k[y_{ij} : 1 \leq i \leq m-r, 1 \leq j \leq q]$ with indeterminates y_{ij} which are linear combination of $z_i, u_{i1}, \cdots, u_{i,q-1}$. Here $1 \leq s \leq q$, $I_s(\mathcal{M}_0)$ and $I_s(\mathcal{N})$ refer to the ideals generated by s-minors of $\mathcal{M}_0$ and $\mathcal{N}$, respectively.

The ideal $I_s(\mathcal{N})$ is a square-free monomial ideal in S and hence a simplicial complex Δ_s is associated to it. It appears that the Alexander dual of Δ_s is a pure shellable complex. Furthermore, the following result may be obtained.

Theorem 3.1. [19, Theorem 4.1] *With the above notation and hypothesis the following hold:*

(a) $\mathrm{depth} S/I_s(\mathcal{N}) = s - 1$.

(b) The ideal $I_s(\mathcal{N})$ has linear quotients. In particular, as an S-module, $S/I_s(\mathcal{N})$ admits a linear resolution and as a ring, it is a Golod ring.

(c) The ring $S/I_s(\mathcal{N})$ is Gorenstein if and only if $m = r+1$ and $s = q$, it is Cohen-Macaulay if and only if $m = r+1$, and for $m \neq r+1$, it is Buchsbaum if and only if $s = 2$.

In fact, a linear resolution of $I_s(\mathcal{N})$ is explicitly constructed in [19]. Furthermore, the Betti numbers of $I_s(\mathcal{N})$ are computed in [18].
The above theorem leads to the following *depth formula* for generic singularities.

Theorem 3.2. [16, Theorem 3.1] *For a singular point $y \in X'$ of multiplicity q the following holds:*

$$\mathrm{depth} \mathcal{O}_{X',y} = r - (q-1)(m-r-1).$$

Corollary 3.1. *The variety X' is Cohen Macaulay at y if and only if $m = r+1$.*

Indeed, the depth formula is valid at every point of multiplicity q even if X' is not analytically irreducible at y provided that the canonical form of π^* is available at this point. This leads to the following result which is a partial answer to a conjecture of Andreotti-Bombieri-Holm on the weak normality of X' (see [17]).

Theorem 3.3. [16, Theorem 3.3] *The variety X' is weakly normal at all points where π^* has canonical form provided that* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > \frac{m+1}{m-r+1}$.

Remark 3.1. *It may be observed that* $\dim(\Delta_s) = (m-r)(s-1)-1$ *and any two facets of* Δ_s *intersect in a face of dimension at most equal to* $(m-r)(s-2)-1$*. Moreover, there are facets which intersect in a face of this maximal dimension. Thus the simplicial complexes* Δ_s *provides natural examples of* CM_τ *complexes for* $\tau = (m-r)(s-2)-1$ *in the sense of [3, Definition 2.1 and Example 2.2]. This also means that* Δ_s *is Cohen-Macaulay in codimension* $m-r-1$ *in the sense of Miller-Novik-Swartz [9, Definition 6.3].*[1]

4. Acknowledgments

The original version of this article was presented at the International School and Conference on Dynamical systems, Singularity theory and Foliations, held at Samarkand State University, Samarkand, Uzbekistan during the period 6-21 October, 2009. The author would like to thank the organizing committee, CIMPA and ICTP for the financial support. He particularly thanks Professor Fouad Elzein for his hard work on this program and Professor Ahmadjon Soleev for his warm hospitality in Samarkand. He is also grateful to the referee for useful remarks, in particular, for bringing attention to the work of Morin [10].

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